Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(04)00028-0

J. Appl. Maths Mechs, Vol. 68, No. 2, pp. 183–198, 2004

0021-8928/$—see front matter

A NEW APPROACH TO THE THEORY OF THE STABILITY OF LINEAR CANONICAL SYSTEMS OF DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS†

A. A. ZEVIN

Dnepropetrovsk

e-mail: zevin@npkista.dp.ua

(*Received* 18 *August* 2002)

The main element of the proposed approach to constructing a theory of the stability of canonical systems is an index function (defined below), which contains all the necessary information on the system. The fundamental results of existing theory, in particular, the necessary and sufficient condition for strong stability, are expressed in new terms. The corresponding proofs only use simple mathematical means; moreover, they are much shorter than existing proofs. A number of new assertions are established, in particular, a simple sufficient condition for strong stability is obtained, which essentially generalizes the well-known Yakubovich theorem [1] of the directed width of the stability regions, and the necessary and sufficient condition for their directed convexity is obtained. Using them, certain non-local qualitative results on the regions of stability of parametric oscillations of canonical systems are established (which enable, in particular, the existing practice of constructing stability regions in accordance with their boundaries to be justified), and the conditions for high-frequency parametric stabilization of unstable systems are obtained.

The theory of the stability of linear canonical systems with periodic coefficients, which had its origin in the publications of Lyapunov [2] and Poincaré [3], has found numerous applications in mechanics, the theory of automatic control, problems of the dynamic stability of elastic systems and other areas of science and technology. The basis of the modern theory is the division of the multipliers into genera, introduced by Krein [4], and the Gel'fand–Lidskii theorem on the structure of the stability regions [5]. Unfortunately, the proofs of many of the theorems are extremely laborious and use quite complex mathematical apparatus, which make them inaccessible for researchers and developers. As a result, when analysing specific systems, only constructive methods (numerical or asymptotic) are usually employed, and the remarkable qualitative results of the theory remain largely unrecognized. Nevertheless, it is precisely the qualitative results that provide the greatest understanding of the problem; they often enable one to draw interesting conclusions regarding the stability of a system even in those cases when its parameters are only known approximately (and when constructive methods are practically useless).

1. THE MAIN IDEAS AND DEFINITIONS

Consider a system of $2n$ linear differential equations of the form

$$J\dot{\mathbf{x}} = H(t)\mathbf{x}, \quad \mathbf{x} \in R^{2n}$$
$$H(t) = H(t+T) = \|h_{ik}(t)\|_{i,k=1}^{2n}, \quad J = \begin{Vmatrix} 0 & -I_n \\ I_n & 0 \end{Vmatrix} \tag{1.1}$$

where I_n is the identity matrix of order n, and $H(t)$ is a symmetrical real piecewise-continuous T-periodic matrix. Before describing the results we will recall some fundamental ideas and definitions, relating to Eq. (1.1).

System (1.1) is said to be stable if all its solutions are bounded as $t \to \infty$. Strongly stable systems, which retain their stability for small perturbations of the Hamiltonian $H(t)$, are of interest for

†*Prikl. Mat. Mekh.* Vol. 68, No. 2, pp. 206–224, 2004.

applications. In a mathematical formulation this means that, in a strongly stable system, there is an $\varepsilon > 0$ such that Eq. (1.1) with any symmetric matrix $H_1(t)$ is also strongly stable, so long as $|H_1(t) - H(t)| < \varepsilon$, where $|A|$ is the norm of the matrix A.

Suppose $X(t)$ is a matrix, the columns of which consist of $2n$ linearly independent solutions of Eq. (1.1). The eigenvalues ρ_k $(k = 1, \dots, 2n)$ of the matrix $X(T)$ are called multipliers of the system. To each simple multiplier ρ_k there corresponds a solution of the form

$$\mathbf{x}_k(t) = \exp(\alpha_k t)\mathbf{f}_k(t); \quad \alpha_k = (\ln\rho_k)/T, \quad \mathbf{f}_k(t+T) = \mathbf{f}_k(t) \tag{1.2}$$

where α_k are characteristic exponents of the system.

If all the multipliers are simple, system (1.1) has $2n$ linearly independent solutions of the form (1.2). The same situation also arises in the case of multiple multipliers, if the elementary divisors of the matrix $X(T)$ are simple (i.e. the number of eigenvectors of the matrix $X(T)$, corresponding to each multiplier ρ_k is equal to its multiplicity r_k when considered as a root of the characteristic equation $\det\|X(T) - \rho I_{2n}\| = 0$). In the case of a multiple multiplier ρ_k with non-simple elementary divisors, in addition to the solution of the form (1.2) there are solutions of the form

$$\mathbf{x}_k(t) = \exp(\alpha_k t)\mathbf{P}_k(t) \tag{1.3}$$

where $\boldsymbol{P}_k(t)$ are polynomials with T-periodic coefficients.

Since Eq. (1.1) is real, in addition to complex multipliers ρ_i there is also a conjugate multiplier ρ_i^*. If $|\rho_i| \neq 1$, there is also a multiplier $1/\rho_i$ (the Lyapunov–Poincaré theorem). It follows from expressions (1.2), (1.3) and the Lyapunov–Poincaré theorem, that all these solutions are bounded as $t \to \infty$ (i.e. the system is stable), provided all the multipliers lie on the unit circle and all the elementary divisors of the matrix $X(T)$ are simple.

For any solutions $\mathbf{x}_i(t)$ and $\mathbf{x}_k(t)$ of Eq. (1.1), we have the identity

$$(\mathbf{x}_i(t), J\mathbf{x}_k(t)) \equiv c_{ik} = \text{const} \tag{1.4}$$

where $(\mathbf{a}, \mathbf{b})$ is the scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

By virtue of expressions (1.2) and (1.3) the left-hand side of identity (1.4) contains the factor $\exp[\alpha_i + \alpha_k^*)t]$, and hence this is possible provided

$$(\mathbf{x}_i(t), J\mathbf{x}_k(t)) = 0 \text{ when } \alpha_i + \alpha_k^* \neq 0 \tag{1.5}$$

As follows from identity (1.4), any matrix of the solutions of the canonical equation satisfies the relation

$$X(t)'JX(t) = C = \|c_{ik}\|_{i,k=1}^{2n} \tag{1.6}$$

where the prime denotes transposition.

For a real non-singular matrix $X(t)$ the inverse assertion holds [5], namely, if the matrix $X(t)$ satisfies relation (1.6), it is a solution of a certain canonical system (1.1) with Hamiltonian

$$A(t) = J\dot{X}(t)X^{-1}(t) \tag{1.7}$$

We will show that this conclusion also holds for the matrix $X(t)$, also containing pairwise conjugate complex columns $\mathbf{x}_k(t) = \mathbf{x}^*{}_{k+1}(t) = \mathbf{u}_k(t) + i\mathbf{v}_k(t)$. In fact, compiling the matrix $X_1(t)$ from the real columns $\mathbf{x}_i(t)$ and the functions $\mathbf{u}_k(t)$ and $\mathbf{v}_k(t)$, we obtain that it satisfies relation (1.6) and is therefore a solution of the canonical equation with Hamiltonian $A_1(t) = J\dot{X}_1(t)X_1^{-1}(t)$. But the matrix $X(t)$, being a linear combination of the columns of $X_1(t)$, also serves as a solution of this equation, and hence $A_1(t) = J\dot{X}(t)X^{-1}(t)$.

2. PRELIMINARY RESULTS

We will first establish some subsidiary results. Consider the boundary-value problem

$$\begin{aligned} &J\dot{\mathbf{x}} + 2\pi mT^{-1}\mathbf{x} = \lambda R(t)\mathbf{x}, \quad \mathbf{x}(T) = \exp(i\varphi)\mathbf{x}(0) \\ &R(t) = H(t) + 2\pi mT^{-1}I_{2n} \end{aligned} \tag{2.1}$$

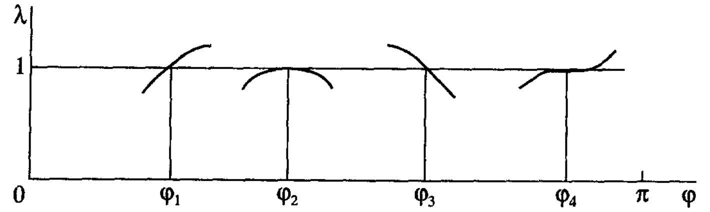

Fig. 1

where m is an integer such that $R(t) > 0$ when $t \in [0, T]$. It is obvious that this inequality is satisfied if $m > -h(t)T/(2\pi)$, where $h(t)$ is the least eigenvalue of the matrix $H(t)$. If $H(t) > 0$, then $h(t) > 0$, and we can take $m = 0$, in which case $R(t) = H(t)$.

Since the matrix $R(t)$ is symmetrical, problem (2.1) is self-conjugate; by virtue of the fact that $R(t) > 0$ its eigenvalues λ_i ($i = 1, 2, \ldots$) are real, and the corresponding eigenfunctions satisfy the relation [1]

$$\int_0^T (R(t)\mathbf{x}_i(t), \mathbf{x}_k(t))dt = 0 \quad \text{when} \quad \lambda_i \neq \lambda_k \tag{2.2}$$

Boundary condition (2.1) depends analytically on φ, and hence $\lambda_i(\varphi)$ and the eigenfunctions $\mathbf{x}_i(t, \varphi)$ are analytic in φ.

When $\lambda = 1$ Eq. (2.1) is identical with (1.1). Hence, if $\lambda_i(\varphi_k) = 1$ for certain i and φ_k, Eq. (1.1) has the multiplier $\rho = \exp(i\varphi_k)$ (which can also be multiple, even if the eigenvalue $\lambda_i(\varphi_k)$ is simple). Hence, points at which the graphs of the functions $\lambda_i(\varphi)$ intersect or are tangent to the straight line $\lambda = 1$ indicate the position of the multipliers of Eq. (1.1) on the unit circle. Thus, in the situation represented in Fig. 1, four multipliers lie on the upper semicircle (as follows from later results, multipliers at the points φ_1 and φ_3 are simple, and at the point φ_2 and φ_4 they are multiple).

Suppose $\rho_k = \exp(i\varphi_k)$ is a multiplier of multiplicity $r \geqslant 1$ with simple elementary divisors. Then boundary-value problem (2.1) when $\varphi = \varphi_k$ has an r-tuple eigenvalue $\lambda = 1$. Suppose $\lambda_p(\varphi)$ ($p = 1, \ldots, r$) are the corresponding analytic functions ($\lambda_p(\varphi_k) = 1$); we will put $\lambda_{p\varphi}(\varphi_k) = d\lambda_p(\varphi)/d\varphi|_{\varphi = \varphi_k}$.

Lemma 1. The following inequality holds

$$\lambda_{p\varphi}(\varphi_k) \neq 0, \quad p = 1, \ldots, r \tag{2.3}$$

Proof. The eigenfunction $\mathbf{x}_p(t, \varphi) = \exp(i\varphi t/T)\mathbf{f}_p(t, \varphi)$ corresponds to the eigenvalue $\lambda_p(\varphi)$ ($p = 1, \ldots, r$), where $\mathbf{f}_p(t, \varphi) = \mathbf{f}_p(t + T, \varphi)$. Substituting $\mathbf{x}_p(t, \varphi)$ into Eq. (2.1), we obtain

$$J\dot{\mathbf{f}}_p + 2\pi m T^{-1}\mathbf{f}_p = \lambda_p R\mathbf{f}_p - i\varphi T^{-1}J\mathbf{f}_p \tag{2.4}$$

Differentiating relation (2.4) with respect to φ and taking into account the equation $\lambda_p(\varphi_k) = 1$, we obtain that $\mathbf{f}_{p\varphi}(t) = \partial\mathbf{f}_p(t, \varphi)/\partial\varphi|_{\varphi = \varphi_k}$ satisfies the equation

$$J\dot{\mathbf{f}}_{p\varphi} + 2\pi m T^{-1}\mathbf{f}_{p\varphi} = R\mathbf{f}_{p\varphi} - i\varphi_k T^{-1}J\mathbf{f}_{p\varphi} + \mathbf{p}(t) \tag{2.5}$$

where

$$\mathbf{p}(t) = -iT^{-1}J\mathbf{f}_p(t, \varphi_k) + \lambda_{p\varphi}R(t)\mathbf{f}_p(t, \varphi_k)$$

In homogeneous equation (2.5) has a T-periodic solution provided that

$$\int_0^T (\mathbf{p}(t), \mathbf{z}_p(t))dt = 0, \quad p = 1, 2, \ldots \tag{2.6}$$

where $\mathbf{z}_p(t)$ are T-periodic solutions of the homogeneous conjugate equation

$$\dot{\mathbf{z}} = -H(t)J\mathbf{z} - i\varphi_k T^{-1}\mathbf{z} \tag{2.7}$$

It can be shown by direct substitution that Eq. (2.7) has the solutions $\mathbf{z}_p = J\mathbf{f}_p(t)$ ($p = 1, \ldots, r$), and hence from condition (2.6), taking the equality $(\mathbf{f}_p, iJ\mathbf{f}_p) = \text{const}$ into account, we obtain

$$\lambda_{p\varphi} = (\mathbf{f}_p, iJ\mathbf{f}_p)\left(\int_0^T (R\mathbf{f}_p, \mathbf{f}_p)dt\right)^{-1} \tag{2.8}$$

Note that $(\mathbf{f}_p, iJ\mathbf{f}_p)$ is a real number, since

$$(\mathbf{f}_p, iJ\mathbf{f}_p)^* = (iJ\mathbf{f}_p, \mathbf{f}_p) = (\mathbf{f}_p, iJ\mathbf{f}_p)$$

Since the function $\mathbf{x}_p(t, \varphi)$ are continuous in φ, Eq. (2.2) remains true when $\lambda_i(\varphi) = \lambda_k(\varphi)$. Hence, we similarly obtain from relation (2.8)

$$(\mathbf{f}_p, iJ\mathbf{f}_k) = 0, \quad k \neq p, \quad p = 1, \ldots, r \tag{2.9}$$

As follows from condition (1.5), $(\mathbf{f}_p, iJ\mathbf{f}_k) = 0$ when $\alpha_p + \alpha_k^* \neq 0$, and hence Eq. (2.9) holds for all $p \neq k$. Since the function $\mathbf{f}_p$ cannot be orthogonal to the $2n$ linearly independent functions $J\mathbf{f}_k$, when have $(\mathbf{f}_p, iJ\mathbf{f}_p) \neq 0$; consequently $\lambda_{p\varphi} \neq 0$.

It follows from this lemma, for example, that non-simple elementary dividers (corresponding to the products $\lambda_{i\varphi}(\varphi_i) = 0$) correspond to the multipliers at the points φ_2 and φ_4 (Fig. 1).

Remark. As pointed out above, the basis of the existing theory of canonical systems is the division of multipliers into genera, introduced by Krien [4]. If, with condition (2.9) $(\mathbf{f}_p, iJ\mathbf{f}_p) > 0$ ($(\mathbf{f}_p, iJ\mathbf{f}_p) < 0$) for all $p = 1, \ldots, r$, this r-tuple multiplier is called a multiplier of the first (second) kind [5]. As can be seen from expression (2.8), all the derivatives of the functions $\lambda_p(\varphi)$ ($p = 1, \ldots, r$, $\lambda_p(\varphi_k) = 1$) at the point φ_k are then correspondingly positive or negative ($R(t) > 0$). Hence, Lemma 1 gives the division of the multipliers into genera a clear geometrical meaning. However, this classification of the multipliers is not used below, since all the results can be expressed using the function $\lambda_p(\varphi)$.

The following lemma extends Lemma 1 to the case when the multiplier $\rho_k = \exp(i\varphi_k)$ of any multiplicity r corresponds to the simple eigenvalue $\lambda_i(\varphi_k) = 1$. Since we do not use it in what follows when analysing stability, we will present it without proof.

Lemma 2. The following relations hold

$$\lambda_{i\varphi}^p(\varphi_k) = \frac{d^p}{d\varphi^p}\lambda_i(\varphi)|_{\varphi = \varphi_k} = 0, \quad p = 1, \ldots, r-1, \quad \lambda_{i\varphi}^r(\varphi_k) \neq 0 \tag{2.10}$$

Hence, the derivatives of the function $\lambda_i(\varphi)$ characterize the multiplicity of the corresponding multipliers (namely, the multiplicity is equal to the order of the lowest derivative, not equal to zero).

It follows from Lemma 2, in particular, that simple multipliers lie at the points φ_1 and φ_3 (Fig. 1) (the first derivatives of the functions $\lambda_i(\varphi)$ at these points are not equal to zero). The first and second derivatives are equal to zero at the points φ_2 and φ_4 respectively; if the next derivatives are not equal to zero, two multipliers lie at the point φ_2 and the three multipliers lie at the point φ_4.

We will assume that the Hamiltonian $H = H(t, \varepsilon)$ depends analytically on the parameter ε; then, for fixed φ, the eigenvalues of problem (2.1) $\lambda_p = \lambda_p(\varepsilon)$ ($p = 1, 2, \ldots$). Assuming $H = H(t, \varepsilon)$, $\mathbf{x} = \mathbf{x}_p(t, \varepsilon)$, $\lambda = \lambda_p(\varepsilon)$ in (2.1) and differentiating with respect to ε, we obtain

$$J\dot{\mathbf{x}}_{p\varepsilon} + 2\pi m T^{-1}\mathbf{x}_{p\varepsilon} = \lambda_p R\mathbf{x}_{p\varepsilon} + \lambda_{p\varepsilon}R\mathbf{x}_p + \lambda_p H_\varepsilon \mathbf{x}_p \tag{2.11}$$

where

$$H_\varepsilon = \partial H(t, \varepsilon)/\partial\varepsilon, \quad \lambda_{p\varepsilon} = d\lambda_p(\varepsilon)/d\varepsilon, \quad \mathbf{x}_{p\varepsilon} = d\mathbf{x}_p(t, \varepsilon)/d\varepsilon$$

Hence, in the same way as when deriving formula (2.8), we obtain

$$\lambda_{p\varepsilon} = -\lambda_p\int_0^T (H_\varepsilon\mathbf{x}_p, \mathbf{x}_p)dt\left(\int_0^T (R\mathbf{x}_p, \mathbf{x}_p)dt\right)^{-1} \tag{2.12}$$

We will assume that the Hamiltonian increases as ε increases; then $H_\varepsilon(t, \varepsilon) > 0$. By virtue of expression (2.12), the positive eigenvalues $\lambda_p(\varphi, \varepsilon)$ decrease as ε increases. Suppose $\lambda_p(\varphi_k, \varepsilon) = 1$; it is then obvious

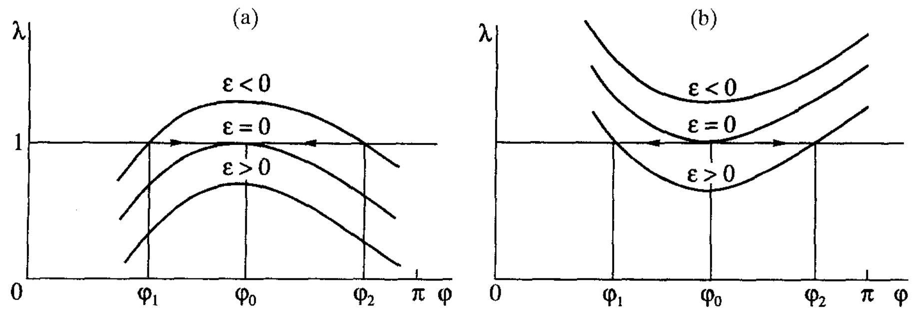

Fig. 2

that if the derivative $\lambda_{p\varphi}(\varphi_k, \varepsilon) > 0$, then $\varphi_k(\varepsilon)$ increases as ε increases (i.e. the multiplier $\rho = \exp(i\varphi_k(\varepsilon))$ moves in an anticlockwise direction); otherwise, if $\lambda_{p\varphi}(\varphi_k, \varepsilon) < 0$, $\varphi_k(\varepsilon)$ decreases.

Suppose the multipliers $\rho_1 = \exp(i\varphi_1)$ and $\rho_2 = \exp(i\varphi_2)$ correspond to successive roots of the equation $\lambda_p(\varphi, \varepsilon) = 1$, where $\lambda_p(\varphi, \varepsilon) > 1$ when $\varphi \in (\varphi_1, \varphi_2)$ (Fig. 2a). Then these multipliers move along the unit circle in opposite directions as ε increases; if, for a certain $\varepsilon = \varepsilon_0$, they meet at a certain point φ_0, non-simple elementary divisors will correspond to the corresponding double multiplier ρ_0 (the eigenvalue $\lambda_p(\varphi_0, \varepsilon_0) = 1$ is simple, and hence one eigenvector will correspond to the eigenvalue ρ_0 of the matrix $X(T)$). It is obvious that when ε increases further in a fairly small neighbourhood of the point φ_0 the equation $\lambda_p(\varphi, \varepsilon) = 1$ has no roots, i.e. the multipliers considered converge with the circle.

In Fig. 2(b) the function $\lambda_p(\varphi, \varepsilon)$ is convex downwards with respect to φ. When $\varepsilon = \varepsilon_0$ it touches the straight line $\lambda = 1$ at the point φ_0; this indicates that certain multipliers fall on the circle (by virtue of Lemma 3 the number of such multipliers r is equal to the order of the lowest non-zero derivative of the function $\lambda_p(\varphi, \varepsilon_0)$ when $\varphi = \varphi_0$; in the case of a common position $r = 2$). When ε increases further, the curve of $\lambda_p(\varphi, \varepsilon)$ intersects the straight line $\lambda = 1$ at two points, where the corresponding derivatives are not equal to zero. Hence, in accordance with Lemma 3 simple multipliers lie at these points, i.e. $r - 2$ of the coinciding multipliers again converge with the circle.

It is clear that if the eigenvalue $\lambda(\varphi_0, \varepsilon_0) = 1$ is multiple, the behaviour of the multipliers in the neighbourhood of φ_0 is determined by each of the corresponding functions $\lambda_p(\varphi, \varepsilon)$ separately.

Note that these results agree completely with the results on the motion of multipliers when the Hamiltonian increases, obtained by Krein and Lyubarskii using different considerations [6].

3. THE INDEX AND INDEX FUNCTION OF A SYSTEM

Suppose $N(\varphi)$ is the number of eigenvalues of problem (2.1) in the range (0, 1). The following ideas [7, 8] play a key role in the theory.

Definition. We will call the function

$$q(\varphi) = N(\varphi) - 2mn \tag{3.1}$$

the index function, and the number $q = q(0) = N(0) -$ the $2mn$ index of the Hamiltonian $H(t)$.

We first note that the index is independent of m, so long as the inequality $R(t) = H(t) + 2\pi m T^{-1} I_{2n} > 0$ is satisfied. The point is that [7, 8], when m increases by unity the number $N(\varphi)$ increases by $2n$, and hence the value of q remains unchanged.

When $\lambda = \lambda_p(\varphi)$ the corresponding solution $\mathbf{x}_p(t, \varphi)$ satisfies boundary condition (2.1), and hence Eq. (2.1) has the multiplier $\rho = \exp(i\varphi)$. Since there is then also a conjugate multiplier $\rho^* = \exp(-i\varphi)$, we have $\lambda_p(\varphi) = \lambda_p(-\varphi)$. Consequently, $q(\varphi) = q(-\varphi)$, i.e. $q(\varphi)$ is an even function and it is therefore sufficient to consider it solely in the range $[0, \pi]$.

It is obvious that the function $q(\varphi)$ is piecewise-constant, and discontinuities can only occur at those points φ_k at which a certain eigenvalue $\lambda_p(\varphi_k) = 1$ or $\lambda_p(\varphi_k) = 0$. When $\lambda = 0$ all the multipliers of Eq. (2.1) are equal to unity, and hence $\lambda_p(\varphi) \neq 0$ when $\varphi \in (0, \pi)$, i.e. discontinuities can only occur here at those points φ_k at which there is a multiplier $\rho_k = \exp(i\varphi_k)$ $(\lambda_p(\varphi_k) = 1)$.

Suppose the eigenvalue $\lambda(\varphi_k) = 1$ has multiplicity r, and p of the functions $\lambda_p(\varphi)$ at this point decrease and $r - p$ increase. It is obvious that the increment of the index function at this point

$$\Delta q(\varphi_k) = q(\varphi_k + 0) - q(\varphi_k - 0) = p - r \tag{3.2}$$

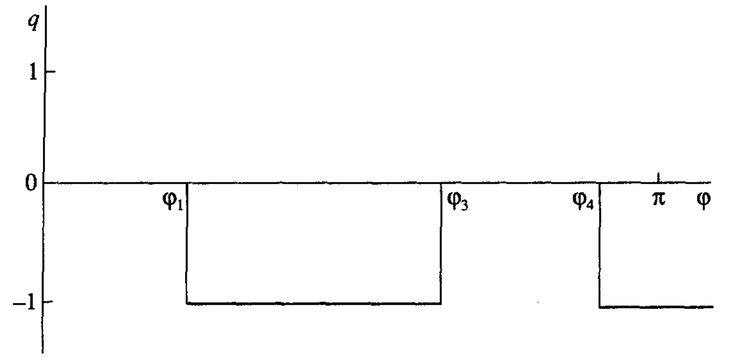

Fig. 3

When $\varphi \to 0$, n positive eigenvalues $\lambda_i(\varphi) \to 0$, and hence the function $N(\varphi)$ is discontinuous at the point $\varphi = 0$, even if $\lambda_p(0) \neq 1$. Here $N(+0) - N(0) = n$ and, consequently, the index $q = q(+0) - n$.

It Fig. 3 we show, as an example, a graph of the index function corresponding to the functions $\lambda_i(\varphi)$, presented in Fig. 1; it was assumed here that $q = -n$, and therefore $q(+0) = 0$.

As is well known, the solution of Eq. (2.1) with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ can be represented in the form $\mathbf{x}(t) = W(t, \lambda)\mathbf{x}_0$, where $W(t, \lambda)$ is the matrix of solutions which satisfy the condition $W(0, \lambda) = I_{2n}$ (the matricant). Hence, the eigenvalues $\lambda_i(\varphi)$ $(i = 1, 2, \ldots)$ of problem (2.1) are the roots of the equation $\det\|W(T, \lambda) - \exp(i\varphi)I_{2n}\| = 0$. It can be shown that if $H(t) > 0$, the number $N(\varphi)$ is equal to the number of zeros $t_i \in (0, T)$ (taking their multiplicity into account) of the equation $\det\|W(t, 1) - \exp(i\varphi)I_{2n}\| = 0$. This result considerably simplifies the calculation of the index function.

The following result will often be used later.

Lemma 3. The index function $q(\varphi)$ increases (does not decrease) when the Hamiltonian increases.

In fact, by virtue of (2.12), when the Hamiltonian increases the positive eigenvalues of problem (2.1) decrease, and hence the Number $N(\varphi)$ of eigenvalues in the range (0, 1), and together with them the index function also, can only increase.

It follows from Lemma 3, for example, that when the Hamiltonian increases, the point φ_3 of the index function (Fig. 3) moves in the negative direction, while the points φ_1 and φ_4 move in the positive direction.

4. THE NECESSARY AND SUFFICIENT CONDITION FOR STRONG STABILITY

Suppose stable system (1.1) is not strongly stable. Then, by definition, a Hamiltonian, continuous in ε, exists, such that, for as a small a value of $\varepsilon > 0$ as desired, the system is unstable, i.e. a certain multiplier $\rho_i(\varepsilon)$ does not lie on the unit circle. Since in this case a multiplier $\rho_q(\varepsilon) = 1/\rho_i^*(\varepsilon)$ exists, we have $\lim\rho_q(\varepsilon) = \lim\rho_i(\varepsilon) = \rho_i(0)$ as $\varepsilon \to 0$. Hence, this situation is possible only if the multiplier $\rho_i(0)$ is multiple. Consequently, the stable system (1.1) is strongly stable if all the multipliers are simple.

As Krien showed [4], this sufficient condition, in general, is not necessary. The necessary and sufficient conditions for strong stability are established by the Krien–Gel'fand–Lidskii theorem [1]. This theorem will be proved below in terms which differ from the classical ones.

We will assume that Eq. (1.1) is stable. Suppose φ_k $(k = 1, \ldots, l \leqslant n)$ are the arguments of the multipliers on the upper semicircle ($l = n$, if all the multipliers are simple).

Theorem 1. For strong stability of Eq. (1.1) it is necessary and sufficient that

$$K = \sum_{k=1}^{l} |\Delta q(\varphi_k)| = n, \quad 0 < \varphi_k < \pi \tag{4.1}$$

Proof. Equation (4.1) obviously implies that on the upper (and, consequently, on the lower) semicircle there are n multipliers, and simple elementary dividers correspond to multiple multipliers (as was shown above, these conditions guarantee stability). When condition (4.1) holds, the functions $\lambda_i(\varphi)$ intersect the straight line $\lambda = 1$ n times at $(0, \pi)$, and if $\lambda_i(\varphi_k) = \ldots = \lambda_{i+r-1}(\varphi_k) = 1$ at a certain point φ_k, then all the derivatives $\lambda_{p\varphi}(\varphi_k)$ $(p = i, \ldots, i+r-1)$ have the same signs (otherwise $|\Delta q(\varphi_k)| < r$ and condition (4.1) is not satisfied). Hence, taking into account the continuity of the Hamiltonian $H(t, \varepsilon)$ and, consequently, of the functions $\lambda_i(\varphi, \varepsilon)$ with respect to ε, it follows that for sufficiently small ε the number of intersections remains equal to n, i.e. stability is maintained. Hence, the sufficiency of condition (4.1) is proved. We will show that this is necessary.

Suppose condition (4.1) is not satisfied in the stable system $(K < n)$. This implies that at a certain point φ_k there are derivatives $\lambda_{i\varphi}(\varphi_k)$ with different signs and either $\lambda_p(0) = 1$ or $\lambda_p(\pi) = 1$ (i.e. there is a multiplier $\rho = 1$ or $\rho = -1$). In the latter cases, at the point $\varphi = 0$ or $\varphi = \pi$, there are also derivatives $\lambda_{i\varphi}$ of different signs ($q(\varphi) = q(-\varphi)$ and $q(\varphi) = q(2\pi - \varphi)$). We will show that when these derivatives are present system (1.1) is strongly stable.

The solutions of the stable system can be represented in the form

$$\mathbf{x}_k(t) = \exp(i\varphi_k t/T)\mathbf{f}_k(t), \quad \mathbf{f}_k(t+T) = \mathbf{f}_k(t), \quad k = 1, \ldots, 2n \tag{4.2}$$

where $\mathbf{x}_k(t)$ is the eigenfunction of problem (2.1) when $\varphi = \varphi_k$, corresponding to the eigenvalue $\lambda = 1$. We will assume, without loss of generality, that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ correspond to the eigenfunctions $\lambda_1(\varphi)$ and $\lambda_2(\varphi)$, where $\lambda_1(\varphi_1) = \lambda_2(\varphi_1) = 1$, $\lambda_{1\varphi}(\varphi_1) > 0$, $\lambda_{2\varphi}(\varphi_1) < 0$. Then, by virtue of relation (2.8), we can assume

$$(\mathbf{x}_1, iJ\mathbf{x}_1) = (\mathbf{f}_1, iJ\mathbf{f}_1) = 1 \quad (\mathbf{x}_2, iJ\mathbf{x}_2) = (\mathbf{f}_2, iJ\mathbf{f}_2) = -1 \tag{4.3}$$

Suppose $X(t) = [\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_{2n}(t)]$ is the matrix of the solutions of Eq. (1.1), where $\mathbf{x}_{k+n}(t) = \mathbf{x}_k^*(t)$. We will put $U(t, \varepsilon) = [\mathbf{u}_1(t, \varepsilon), \ldots, \mathbf{u}_{2n}(t, \varepsilon)]$, where

$$\mathbf{u}_1 = \mathbf{u}_{n+1}^* = \exp(\varepsilon t)[\mathbf{x}_1(t) + \mathbf{x}_2(t)], \quad \mathbf{u}_2 = \mathbf{u}_{n+2}^* = \exp(-\varepsilon t)[\mathbf{x}_1(t) - \mathbf{x}_2(t)] \tag{4.4}$$

and the remaining functions $\mathbf{u}_i = \mathbf{x}_i(t)$.

By virtue of relations (4.3) and (4.4)

$$\begin{aligned}(\mathbf{u}_1, iJ\mathbf{u}_1) &= \exp(2\varepsilon t)[(\mathbf{f}_1, iJ\mathbf{f}_1) + (\mathbf{f}_2, iJ\mathbf{f}_2) = 0\\ (\mathbf{u}_2, iJ\mathbf{u}_2) &= \exp(-2\varepsilon t)[(\mathbf{f}_1, iJ\mathbf{f}_1) + (\mathbf{f}_2, iJ\mathbf{f}_2) = 0\end{aligned} \tag{4.5}$$

The remaining expressions for $(\mathbf{u}_p, iJ\mathbf{u}_q)$ are identical with $(\mathbf{x}_p, iJ\mathbf{x}_q)$ or are equal to zero, since they contain factors $(\mathbf{f}_p, iJ\mathbf{f}_q)$ that are equal to zero, where $p \neq q$. Hence, the matrix $U(t, \varepsilon)$ satisfies relation (1.6). Hence system (1.1) with the Hamiltonian $H(t, \varepsilon) = J\dot{U}(t, \varepsilon)U(t, \varepsilon)^{-1}$ is canonical. Obviously the Hamiltonian $H(t, \varepsilon)$ is continuous in ε and when $\varepsilon = 0$ is identical with $H(t, 0)$ (the functions $\mathbf{u}_i(t, 0)$ are linear combinations of $\mathbf{x}_i(t)$). Since the solution $\mathbf{u}_1(t, \varepsilon)$ is unbounded for any $\varepsilon > 0$ as $t \to \infty$, the system is stable. The theorem is proved.

Suppose, for example, the multiplicity of the multiplier $\rho = \exp(i\varphi_k)$, which occurs in the proof of the theorem, is equal to two. Then the functions $\lambda_1(\varphi)$ and $\lambda_2(\varphi)$ have the form represented in Fig. 4(a). We will put $H(t, \varepsilon) = H(t) + \varepsilon Q(t)$, where $Q(t) > 0$. Since $H(t, \varepsilon)$ increases as ε increases, the eigenvalues $\lambda_p(\varepsilon)$ decrease, and hence when $\varepsilon < 0$ and $\varepsilon > 0$ the graphs of $\lambda_1(\varphi, \varepsilon)$ and $\lambda_2(\varphi, \varepsilon)$ have the form shown in Figs 4(b) and 4(c), respectively (in the case of a common position for a small perturbation the multiple eigenvalue λ_k splits into two simple ones). In both cases the number of roots of the equation $\lambda_p(\varphi, \varepsilon) = 1$ in the neighbourhood of φ_k is equal to two, and consequently, the system remains stable under this perturbation.

We will now assume that the perturbation $\varepsilon Q(t)$ is such that the eigenvalue $\lambda_1(\varphi_k, \varepsilon)$ increases and $\lambda_2(\varphi_k, \varepsilon)$ decreases with ε (this is possible when the matrix $Q(t)$ is not sign definite). Then for small ε neither of the curves $\lambda_1(\varphi, \varepsilon)$ nor $\lambda_2(\varphi, \varepsilon)$ intersect the straight line $\lambda = 1$ (Fig. 4d), i.e. the multipliers considered converge to the unit circle (the perturbation $H(t, \varepsilon)$, constructed in the theorem, belongs exactly to this type).

Suppose $\varphi_i (i = 1, 2, \ldots)$, $\varphi_{i+1} > \varphi_i$ is an arbitrary set of points in $(0, \pi]$, in which the index function $q(\varphi)$ of Eq. (1.1) is continuous. We will put

$$S = \sum |q(\varphi_{i+1}) - q(\varphi_i)| \tag{4.6}$$

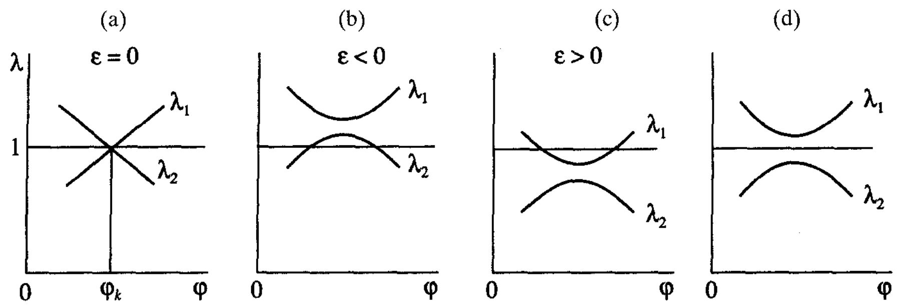

Fig. 4

Corollary 1. If

$$S = n \tag{4.7}$$

then Eq. (1.1) is strongly stable.

The truth of this assertion follows from the obvious fact that $S \leq K \leq n$, and hence Eq. (4.7) guarantees that the stability condition (4.1) is satisfied.

Note that if the points $\varphi_i (i = 1, 2, \ldots)$ correspond to successive extrema of the function $q(\varphi)$, then $S = K$; This assertion will be used later.

In particular, assuming $\varphi_1 = +0$, $\varphi_2 = \pi$ in (4.6), we obtain the following sufficient condition for stability.

Corollary 2. If

$$|q(+0) - q(\pi)| = n \tag{4.8}$$

Eq. (1.1) is strongly stable.

It is obvious that under these conditions the index function decreases monotonically or increases in the range $(0, \pi)$.

Taking into account the fact that

$$q(+0) = N(0) + n - 2mn, \quad q(\pi) = N(\pi) - 2mn$$

We can write condition (4.8) in the form

$$|N(0) + n - N(\pi)| = n \tag{4.9}$$

Note that whereas for a certain stability region condition (4.8) is satisfied, when there is a continuous change in the Hamiltonian it breaks down if and only if there is a T-periodic or T-antiperiodic solution; in this case the strong stability also breaks down. Hence, for this region condition (4.8) is not only sufficient but also necessary.

5. ANALYSIS OF THE REGIONS OF STRONG STABILITY

Strongly stable Hamiltonians $H_1(t)$ and $H_2(t)$ belong to one and the same stability region, if a symmetric matrix $H(t, s) = H(t + T, s)$, piecewise-continuous with respect to t and continuous with respect to s, exists such that $H(t, 0) = H_1(t)$ and $H(t, 1) = H_2(t)$, and when $H = H(t, s)$ Eq. (1.1) is strongly stable for any s ε [0, 1] [1].

The necessary and sufficient condition for $H_1(t)$ and $H_2(t)$ to belong to one and the same stability region was established by Gel'fand and Lidskii [5]; the corresponding theorem is one of the main results of this theory. Nevertheless, it should be noted that the proof of the theorem is extremely laborious and uses fairly complex mathematical apparatus. This important results obtained by Yakubovich on the directed width and convexity of the stability regions [1, 9], the proofs of which are also extremely laborious, also touch on this problem.

Below, this range of problems is solved using the ideas introduced above and the results obtained earlier. This enables us, using the minimum mathematical methods, to give a much more compact description of the theory and to obtain a number of new results.

Suppose $q_1, q_2, \dots, q_{p+1} (q_1 = q(+0), q_{p+1} = q(\pi), p \leq n)$ are successive extrema of the index function in the range $(0, \pi]$, and $\Gamma_1, \dots, \Gamma_{p+1} \in (0, \pi]$ are the corresponding intervals ($q(\varphi) = q_i$ when $\varphi \in \Gamma_i$). The set of integers $\mu = (\mu_1, \dots, \mu_p)$, where $\mu_i = q_{i+1} - q_i$) will be called a multiplier type of Eq. (1.1). It is clear from the results obtained in Section 3 that μ_i is equal to the number of eigenvalues $\lambda_i(\varphi) = 1$ in $(\varphi_i, \varphi_{i+1})$, where all the derivatives of $\lambda_{i\varphi}(\varphi)$ are of one sign (negative when $\mu_i > 0$ and positive when $\mu_i < 0$). For example, for the index function presented in Fig. 3, we have $q_1 = 0$, $q_2 = -1$, $q_3 = 0$, $q_4 = -1$, and hence $\mu = (-1, 1, -1)$.

It is obvious that the strong convergence condition (4.1) is equivalent to the equality

$$M = \sum_{k=1}^{p} |\mu_k| = n \tag{5.1}$$

This condition is satisfied by 2^n different multiplier types [1].

When there is a continuous change in the Hamiltonian the multiplier type changes if and only if a certain interval Γ_i contracts to a point ($\varphi_i = \varphi_{i+1}$). It is obvious that in this case the value of M decreases, and hence condition (5.1) breaks down. Consequently, in order that the Hamiltonians $H_1(t)$ and $H_2(t)$ should belong to one and the same stability region it is necessary that their multiplier types μ^1 and μ^2 should be identical. This condition is not sufficient; the additional condition is established by the following Gel'fand–Lidskii theorem [5].

Theorem 2. In order for the Hamiltonians $H_1(t)$ and $H_2(t)$ with the same multiplier type to belong to one and the stability region, it is necessary and sufficient that their indices should be equal.

In the original proof of this theorem [5] another definition of the index of a Hamiltonian was used. The proof is considerably simplified if we use this index q (see [7]). Note also that the necessity of the condition $q_1 = q_2$ follows directly from the definition of the index. In fact, if a strongly stable curve $H(t, s)$ ($H(t, 0) = H_1(\mathrm{t}), H(t, 1) = H_2(t)$) exists, the multipliers $\rho_i(s) \neq 1$. Consequently, when $\varphi = 0$ the eigenvalues of problem (2.1) $\lambda_i(s) \neq 1$, and as a result the number of eigenvalues in range (0, 1), and together with it the index $q(s)$ also, remains constant when $s \in [0. 1]$.

Hence, the stability regions are determined by the multiplier type and the index; we will denote them by G_μ^q [1].

If $\lambda_i(\varphi) = 1$ and $\lambda_{i\varphi}(\varphi) > 0$ in problem (2.1), then, by virtue of the evenness of the index function $q(\varphi)$, an eigenvalue $\lambda_i(-\varphi) = 1$ exists, where $\lambda_{i\varphi}(-\varphi) < 0$. Hence, if a pair of multipliers passes through the point $\rho = 1$, moving along the unit circle, the number of eigenvalues $\lambda_i \in (0, 1)$ is changed by two. Taking this into account, it is easy to show that the index of a strongly stable Hamiltonian $q = 2k$, where k is an integer (it is equal to the Gel'fand–Lidskii index). It can also be shown that for any Hamiltonian with index q one can construct a Hamiltonian with the same multiplier type and index $q_i = q + 2i$, where i is an arbitrary integer.

In practice, the Hamiltonian $H(t)$ is often not known exactly; in particular, in many cases one can only indicate its bilateral limits, i.e.

$$H_-(t) \leq H(t) \leq H_+(t) \tag{5.2}$$

Suppose that, when $H = H_-(t)$ and $H = H_+(t)$. Equation (1.1) belongs to one stability region G_μ^q. Yakubovich's theorem on the directed width of the stability regions [1] asserts that if Eq. (1.1) with Hamiltonian

$$H(t, s) = H_-(t) + s(H_+(t) - H_-(t))$$

is strongly stable for all $s \in [0, 1]$, it is stable for any $H(t)$ which satisfies condition (5.2).

The following theorem enables us to establish the stability of system (1.1), (5.2) directly from the index functions $q_-(\varphi)$ and $q_+(\varphi)$ of the Hamiltonians $H_-(t)$ and $H_+(t)$ without any additional calculations.

Suppose Γ_i^- and Γ_i^+ ($i = 1, \dots, p + 1$) are the above-mentioned intervals in which the functions $q_-(\varphi)$ and $q_+(\varphi)$ have local extrema. Since, by our conditions, the indices and multiplier types of the systems considered are the same, we have $q_-(\Gamma_i^-) = q_+(\Gamma_i^+)$, but Γ_i^- and Γ_i^+, generally speaking, cannot have common points. We will assume, to fix our ideas, that $\mu_1 = q_2 - q_1 > 0$, in which case Γ_1, Γ_3, Γ_5, ... correspond to the minima and Γ_2, Γ_4, Γ_6, ... correspond to the maxima of the index function.

Theorem 3. If Γ_i^- and Γ_i^+ ($i = 1, 3, \dots$) have common points φ_i, system (1.1), (5.2) is strongly stable.

Proof. Suppose φ_i ($i = 2, 4, \ldots$) are any points from the ranges of Γ_i^-. Since $\varphi_1, \varphi_2, \varphi_3, \ldots$ are sequential extrema of the index function $q_-(\varphi)$ of the strongly stable Hamiltonian $H_-(t)$, we have

$$S_- = \sum_{i=1}^{p} |q_-(\varphi_{i+1}) - q_-(\varphi_i)| = n$$

We will show that the equality $q_-(\varphi_i) = q_+(\varphi_i)$ also holds for even i. In fact, $q_-(\varphi_i) \leqslant q_+(\varphi_i)$ by virtue of the fact that $H_-(t) \leqslant H_+(t)$; on the other hand, if $q_-(\varphi_i) < q_+(\varphi_i)$ for certain even i, then

$$S_+ = \sum_{i=1}^{p} |q_+(\varphi_{i+1}) - q_+(\varphi_i)| > n$$

which is impossible. Since, for condition (5.2) $q_-(\varphi) \leqslant q(\varphi) \leqslant q_+(\varphi)$, for the function $q(\varphi)$ for the same φ_i we have $S = n$, i.e. Eq. (1.1) is strongly stable. The theorem is proved.

Remark 1. As can be seen from the proof, under the conditions indicated, the intervals Γ_i^- and Γ_i^+ ($i = 1, \ldots, p + 1$) have common points. Hence, it follows from the inequality $q_-(\varphi_i) \leqslant q_+(\varphi_i)$ that $\Gamma_i^+ \in \Gamma_i^-$ ($i = 1, 3, \ldots$) and $\Gamma_i^- \in \Gamma_i^+$ ($i = 2, 4, \ldots$).

Remark 2. The condition of the theorem is necessary in the sense that, if it is not satisfied, an unstable Hamiltonian $H(t)$ exists which satisfies inequality (5.2). In fact, suppose $H(t, s)$ increases as s, and $H(t, 0) = H_-(t)$, $H(t, 1) = H_+(t)$, $\Gamma_i(s)$ ($i = 1, 3, \ldots$) are intervals corresponding to the minima of the index function $q(\varphi, s)$. Since $q(\varphi, s)$ does not decrease with s, we have $\Gamma_i(s) \in \Gamma_i^-$ for small s. The absence of common points on Γ_i^+ and Γ_i^- denotes that, for certain $s_* < 1$ the interval of $\Gamma_i(s)$ contracts to a point, and hence the corresponding equation (1.1) is not strongly stable (nevertheless, as shown below, when $H = H_+(t)$ it can again belong to the same stability region).

The stability regions G_μ^q for which, for any $H_-(t)$, $H_+(t) \in G_\mu^q$, it follows from inequality (5.2) that $H(t) \in G_\mu^q$, are said to be directionally convex [1]. The sufficient condition for directed convexity was obtained by Yakubovich [9]; in the terms employed in the present paper it denotes that the corresponding multiplier type μ contains no more than two numbers (which, in particular, is necessarily satisfied when $n = 1$ and $n = 2$).

The following theorem gives the necessary and sufficient condition for directed convexity of the stability regions (note that the assertion on the directed convexity of all the stability regions, made in [7], is incorrect).

Theorem 4. For the directed convexity of the region G_μ^q it is necessary and sufficient that

$$|\mu_i| > |\mu_{i-1}| \text{ or } |\mu_i| > |\mu_{i+1}|, \quad i = 2, \ldots, p-1 \tag{5.3}$$

Proof. Condition (5.3) indicates that the sequence $|\mu_i|$, ($i = 1, \ldots, p$) increases when $i = 1, \ldots, k$ and decreases when $i = k + 1, \ldots, p$, where $k \in [1, \ldots, p)$.

Suppose $H_-(t)$ and $H_+(t)$ belong to the regions M_q^i which satisfy condition (5.3). We will first assume that $\mu_1 > 0$, k is odd or $\mu_1 < 0$, k is even; then

$$q_{k+1} = \sum_{r=1}^{k} \mu_r + q > q_i, \quad i \neq k+1$$

The functions $q_-(\varphi)$ and $q_+(\varphi)$ satisfy the inequality $q_-(\varphi) \leqslant q_+(\varphi)$ and have the same maximum q_k; consequently, a point φ_{k+1} is obtained at which $q_-(\varphi_{k+1}) = q_+(\varphi_{k+1}) = q_{k+1}$. When $\varphi < \varphi_{k+1}$, the functions $q_-(\varphi)$ and $q_+(\varphi)$ have the same minimum q_k, and consequently $q_-(\varphi_k) = q_+(\varphi_k) = q_k$ for certain $\varphi_k < \varphi_{k+1}$. Repeating these discussions, we obtain a set of points φ_i ($i = k, k-1, \ldots, 1$ and $i = k + 1, \ldots, p + 1$), at which all the extrema of the functions $q_-(\varphi)$ and $q_+(\varphi)$ coincide. Consequently, the condition of Theorem 3 is satisfied, which guarantees the strong stability of the Hamiltonian $H(t)$ and thereby proves the sufficiency of condition (5.3).

If $\mu_1 < 0$, k is odd or $\mu_1 > 0$, k is even, the proof is exactly the same (here $q_{k+1} < q_i$, $i \neq k + 1$).

We will prove that condition (5.3) is necessary. If it is not satisfied, then the inequality $|\mu_{i-1}| \geqslant |\mu_i| \leqslant |\mu_{i+1}|$ holds for at least one i. We will show that in this case we can construct an unstable Hamiltonian $H(t)$ which satisfies (5.2).

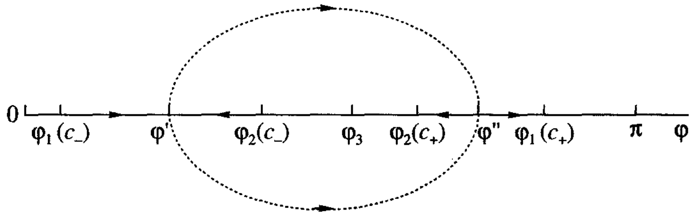

Fig. 5

We will put $H_2(t, c) = H_2(t) + cI_{2n}$, where $H_2(t)$ is a strongly stable second-order Hamiltonian with multiplier type $\mu = (-1, 1)$ (so that the corresponding index function $q(\varphi)$ has a minimum when $\varphi \in (\varphi_i, \varphi_2)$, where φ_1 and φ_2 are the coordinates of the multipliers in the upper semicircle). Since $H_2(t, c)$ and, consequently, $q(\varphi, c)$ increase as c increases, the multipliers $\rho_1(c)$ and $\rho_2(c)$ move in opposite directions (Fig. 5). It is easy to choose $H_2(t, c)$ such that when $c = c'$ they meet at the point φ', converge to the circle and again meet at the point $\varphi'' \in (\varphi', \pi)$ when $c = c''$; they then continue their motion around the circle in the same directions. Hence, for sufficiently small $\varepsilon > 0$ we have $\varphi_2(c'' + \varepsilon) > \varphi_2(c' - \varepsilon)$. It is clear that Eq. (1.1) is unstable when $c \in (c', c'')$ and strongly stable in a certain interval $c \in (c'', c'' + \varepsilon)$, where $\mu(c'' + \varepsilon) = (1, -1)$.

We will add to the system considered a first-order system with constant Hamiltonian

$$H_1(k) = \operatorname{diag}(k, k), \quad kT = \varphi_3 \in (\varphi_2(c_-), \varphi_2(c_+))$$

As a result an additional multiplier $\rho_3 = \exp(i\varphi_3)$ appears in the third-order system obtained with Hamiltonian $H(t, c)$. It is obvious that the Hamiltonians $H_-(t) = H(t, c_-)$ and $H_+(t) = H(t, c_+)$ have the same multiplier type $\mu = (-1, 1, -1)$. Since the corresponding indices are equal (when $c \in [0, c_+]$ none of the multipliers is incident at the point $\rho = 1$), these Hamiltonians belong to one and the same stability region. Hence, the Hamiltonian $H(t, c)$ increases as c increases and belongs to the same stability region when $c = c_-$ and $c = c_+$, but Eq. (1.1) is unstable when $c \in (c', c'') \in (c_-, c_+)$.

The Hamiltonian $H_2(t, c)$ can be chosen in such a way that the indices $H_-(t)$ and $H_+(t)$ have any even value specified in advance. Hence, we can assert that all the stability regions with multiplier type $\mu = (-1, 1, -1)$ are not directionally convex (i.e. those $H_-(t), H_+(t) \in G_\mu^q$ are obtained for which unstable Hamiltonians exist, which satisfy condition (5.2)). The same conclusion also holds for the multiplier type $\mu = (1, -1, 1)$ (here one must consider the Hamiltonian $H_2(t, c)$ for which $\varphi'' < \varphi'$). By combining $|\mu_i|$ systems of this form, we obtain a system of order $n = 3|\mu_i|$ with multiplier type $\mu = (-|\mu_i|, |\mu_i|, -|\mu_i|)$ or $\mu = (|\mu_i|, -|\mu_i|, |\mu_i|)$. It can be extended to the specified system by the addition of first-order systems with multiplier type $\mu_+ = (+1)$ or $\mu_- = (-1)$, the multipliers of which are situated in a corresponding way in the interval $(0, \pi)$. The system constructed has the required multiplier type and is not directionally convex; consequently, condition (5.3) is necessary. The theorem is completely proved.

As can be seen, the directed convexity of the region G_μ^q is defined solely by its multiplier type and is independent of the index.

Since condition (5.3) is independent of $H_-(t)$ and $H_+(t)$, theorem 4 is stronger than Theorem 3 in theoretical respects, but, from the practical point of view, its advantage is small. In fact, as soon as the multipliers of Eq. (1.1) with Hamiltonians $H_-(t)$ and $H_+(t)$ are obtained, the check of the conditions of these theorems is equally elementary.

As pointed out above, in applications hamiltonians of the form $H(t, c_1, \ldots, c_p)$ are usually considered, where c_i are certain parameters. We will denote the regions of strong stability of Eq. (1.1) in the space of these parameters by C_k $(k = 1, 2, \ldots)$. It is obvious that the boundaries of C_k coincide with the boundaries of a certain region G_μ^q, but the latter may contain several regions C_k. Thus, in the single-parameter family $H(t, c)$ considered when proving the theorem, the Hamiltonians $H(t, c_-)$ and $H(t, c_+)$ belong to one stability region G_μ^q, but different regions (sections) of $C_k(c)$ (since when $c \in (c', c'')$ Eq. (1.1) is unstable).

We will assume that the Hamiltonian $H(t, c_1, \ldots, c_p)$ is continuous in all the parameters. We will call the region C_k directionally convex with respect to the parameter c_q if it follows from the condition $H(t, c_1, \ldots, c_q^1, \ldots, c_p), H(t, c_1, \ldots, c_q^2, \ldots, c_p) \in C_k$ that $H(t, c_1, \ldots, c_i, \ldots, c_p) \in C_k$ for all $c_q \in (c_q^1, c_q^2)$.

Theorem 5. If the Hamiltonian $H(t, c_1, \ldots, c_p)$ increases or decreases with respect to the parameter c_q, the stability regions C_k are directionally convex with respect to c_q.

Proof. We will assume the opposite. Then, for certain $c_k (k \neq q)$ we obtain a section $[c_q^1, c_q^2]$ such that $[c_q^1, c_0) \in C_k$, the point c_0 lies on the boundary of region C_k, and the points of the section $[c_0, c_q^2)$ lie inside or on the boundary of C_k. Hence, for any $c \in [c_q^1, c_q^2]$ all the multipliers lie on the unit circle. We can assume, without loss of generality, that the Hamiltonian $H(t, c_1, \ldots, c_p)$ increases with c_p, in which case the index function $q(\varphi, c_q)$ also increases. When $c_q = c_0$ the integral of Γ_i, corresponding to a certain minimum q_i of the function $q(\varphi, c_q)$, contracts to the point φ_i. Suppose that, at the point φ_i, p and q multipliers meet, moving in opposite directions. The corresponding elementary divisors are simple (otherwise, as shown in Section 3, when $c > c_0$ some multipliers will not lie on the circle). Hence, they continue their motion along the circle in the same directions, so that for sufficiently small ε Eq. (1.1) is strongly stable when $c_i \in [c_0 - \varepsilon, c_0)$ and $c_i \in (c_0, c_0 - \varepsilon]$. However, when crossing the point c_0, the multiplier type of the system changes (the index at the point φ_i increases by an amount $p + q$). Hence, the points $c_0 - \varepsilon$ and $c_0 + \varepsilon$ must belong to different stability regions G_μ^q and, consequently, cannot belong to one region C_k. The contradiction obtained proves the theorem.

Hence, unlike the region G_μ^q, all the regions C_k possess the property of directional convexity.

6. THE THEORY OF PARAMETRIC RESONANCE AND PARAMETRIC STABILIZATION

Parametric oscillations of a canonical system are usually described by an equation of the form

$$J\dot{\mathbf{x}} = H(\omega t, \mu)\mathbf{x} \tag{6.1}$$

where $H(\omega t) = H(\omega t + 2\pi)$, ω is the frequency of parametric excitation and μ is a parameter characterizing its intensity. The stability regions in the μ, ω plane represent sets of points, to which strongly stable Hamiltonians $H(\omega t, \mu)$ correspond. On the boundaries of these regions none of the multipliers lie on the unit circle, but Eq. (6.1) is not strongly stable.

There is a vast literature devoted to developing constructive (numerical and analytical) methods for finding the stability regions of Eq. (6.1) (see, for example, [1, 10, 11]). On the other hand, the number of general qualitative assertions regarding the stability regions is limited, and the majority of them were obtained by asymptotic methods assuming the parameter μ to be small [11]. Below we obtain some general results regarding the stability regions, that are free from this limitation. Putting $\tau = \omega t$, we reduce (6.1) to the form

$$\begin{aligned} &J\mathbf{x}' = H(\tau, \mu, \omega)\mathbf{x} \\ &H(\tau, \mu, \omega) = \omega^{-1}H(\tau, \mu), \quad H(\tau, \mu) = H(\tau + 2\pi, \mu) \end{aligned} \tag{6.2}$$

where the prime denotes differentiation with respect to τ.

We will assume that $H(\tau, \mu) > 0$ when $\tau \in [0, 2\pi]$ and $\mu \in [0, \mu_0]$, and $H(\tau, 0) = H_0$ is a constant matrix. Since $H_0 > 0$ the eigenvalues of the matrix $J^{-1}H_0$ are imaginary [1], and we will denote them by $\pm i\omega_k (0 < \omega_1 \leqslant \ldots \leqslant \omega_n)$. When $\mu = 0$ Eq. (6.1) is strongly stable, with the exception of the points [4].

$$\omega = \omega_{pkq} = (\omega_p + \omega_k)/q, \quad p, k = 1, \ldots, n; \quad q = 1, 2, \ldots \tag{6.3}$$

In the case of a common position, the points ω_{pkq} are different. When ω changes at the points $\omega = \omega_{pkq}$ the multiplier type of the system changes, and when $p = k$ and for even q the index of the Hamiltonian also changes (the multipliers pass through the point $\rho = 1$). Hence, the intervals of the ω axis adjoining the point ω_{pkq} correspond to different stability regions G_μ^q and, consequently, to different stability regions in the μ, ω_1 plane (for example, to the regions C_{12} and C_{23} in Fig. 6, where ω^1, ω^2 and ω^3 are neighbouring points of ω_{pkq} on the ω axis).

The following assertion is a direct consequence of Theorem 5.

Corollary 3. If, for a certain μ, the points $\omega'(\mu)$ and $\omega''(\mu)$ belong to one stability region, they also belong to the whole section $[\omega'(\mu), \omega''(\mu)]$.

In fact, the Hamiltonian $H(\tau, \mu, \omega)$ decreases with respect to ω, and hence the stability regions are directionally convex with respect to ω.

The practical importance of this result is as follows. To construct stability regions in the μ, ω plane one usually calculates their boundaries $\omega_i^+(\mu)$ and $\omega_i^-(\mu)$ (Fig. 6), adjoining the points ω_{pkd} of the ω axis; in what follows we will assume that the sets of points between $\omega_i^+(\mu)$ and $\omega_i^-(\mu)$ represent instability

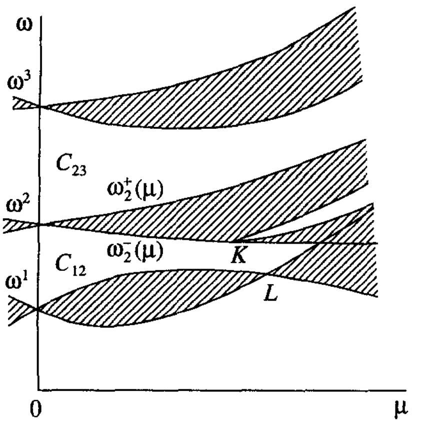

Fig. 6

regions, while the sets of points between $\omega_i^+(\mu)$ and $\omega_{i+1}^-(\mu)$ represent stability regions. This approach needs justification, since, generally speaking, from nowhere does it follow that "islands" of instability lie between the boundaries of the stability regions. Corollary 3 just gives the necessary basis for the case $H(\omega t, \mu) > 0$.

As regards the instability regions, as was shown in [11], for small μ, Eq. (6.1) is in fact unstable for all $\omega \in (\omega_i^-(\mu), (\omega_i^+(\mu))$. A detailed analysis, which is outside the scope of this paper, shows that, in general, stability regions, which adjoin a certain boundary (like, for example, the point K in Fig. 6), lie between these boundaries. Here we have a scenario, described when proving Theorem 4, when a strongly stable equation becomes unstable as the Hamiltonian increases, and then returns to the same stability region.

In applications, one often considers parametric oscillations of systems described by the following second-order vector equation

$$\begin{aligned} &(M(\omega t, \mu)\dot{\mathbf{y}})^{\cdot} + C(\omega t, \mu)\mathbf{y} = \mathbf{0}, \quad y \in R^n \\ &C(\omega t, \mu) = C_0(\mu) + \mu C_1(\mu, \omega t) = C(\omega t + 2\pi, \mu), \quad M(\omega t, \mu) = M(\omega t + 2\pi, \mu) \end{aligned} \tag{6.4}$$

where $M(\omega t, \mu)$ and $C(\omega t, \mu)$ are symmetric matrices, where $M(\omega t, \mu) > 0$ and $C_0(\mu) > 0$, while the average values of the elements of the matrix $C_1(\omega t)$ are equal to zero in the interval $[0, T)$.

By making the replacement $\mathbf{y} = \mathbf{x}_1$, $M\dot{\mathbf{y}} = \mathbf{x}_2$, Eq. (6.4) can be reduced to the form (6.1), where $\mathbf{x} = \mathbf{x}_1, \mathbf{x}_2)$ and $H(\omega t, \mu) = \mathrm{diag}[M^{-1}(\omega t, \mu), C(\omega t, \mu)]$, and hence all the results obtained above remain valid. However, a special form of this Hamiltonian enables us to establish certain additional facts.

First of all, we note that the use of the condition $H(\omega t, \mu) > 0$ above is unnecessary. The fact is that everywhere above the condition $H(t) > 0$ is only necessary to the extent that it guarantees that the following equality is satisfied

$$\int_0^T (H\mathbf{x}, \mathbf{x})dt > 0$$

where $\mathbf{x}(t)$ is the solution of Eq. (1.1), which satisfies the relation $\mathbf{x}(T) = \rho\mathbf{x}(0)$, $|\rho| = 1$. Nevertheless, in the case considered, it is sufficient for this purpose simply for the matrix $M(\omega t, \mu)$ to be positive definite. In fact, taking Eq. (6.4) into account, we obtain

$$\int_0^T (H\mathbf{x}, \mathbf{x})dt = 2\int_0^T (M\dot{\mathbf{y}}, \dot{\mathbf{y}})dt - (M\dot{\mathbf{y}}, \mathbf{y})|_0^T \tag{6.5}$$

By virtue of the relations $\mathbf{y}(T) = \rho\mathbf{y}(0)$, $\dot{\mathbf{y}}(T) = \rho\dot{\mathbf{y}}(0)$, $|\rho| = 1$ the term outside the integral is equal to zero, and as a result the required inequality is satisfied when $\mathbf{y}(t) \neq \mathrm{const}$. Hence, when μ increases the stability regions remain directionally convex with respect to ω, even when the inequality $C(\omega t, \mu) > 0$ breaks down.

Note also that the specific feature of Eq. (6.4) enables a number of assertions on the behaviour of certain boundaries of the stability regions to be established [12, 13].

In conclusion, we will consider the problem of the parameteric stabilization of an unstable system

$$\ddot{\mathbf{y}} + [C_0 + \mu\omega^2 C_1 + \mu\omega^2 C_2(\omega t)]\mathbf{y} = \mathbf{0}, \quad \mathbf{y} \in R^n \tag{6.6}$$

where $C_1 > 0$, while the mean values of the elements of the matrix $C_2(\tau) = C_2(\tau + 2\pi)$ are equal to zero in the range $[0, 2\pi]$.

We will assume that the matrix C_0 is not positive definite. Then Eq. (6.6) is unstable when $\mu = 0$. As is well known, when $\mu \neq 0$ and for sufficiently large ω, Eq. (6.6) may be stable; this effect has come to be called high-frequency parametric stabilization. Existing criteria of the stability of Eq. (6.6) (see, for example, [14, 15]) were obtained by asymptotic methods assuming that the parameter μ is small. Using the results obtained above, we can establish the exact boundaries of the region of parametric stabilization in the μ, ω plane.

Assuming $\tau = \omega t$, we reduce Eq. (6.6) to the form

$$\mathbf{y}'' + [\omega^{-2} C_0 + \mu C_1 + \mu C_2(\tau)]\mathbf{y} = \mathbf{0} \tag{6.7}$$

As was shown in [11], for sufficiently large ω and small μ, Eq. (6.7) belongs to the same stability regions as the equation

$$\mathbf{y}'' + \mu C_1 \mathbf{y} = \mathbf{0} \tag{6.8}$$

We will consider the corresponding boundary-value problems (2.1) ($H = \mathrm{diag}(I, \mu C_1)$, $T = 2\pi/\omega$, $m = 0$ by virtue of the fact that $H > 0$) with periodic boundary conditions ($\varphi = 0$) and antiperiodic boundary conditions ($\varphi = \pi$). The least positive eigenvalues of these problems are equal to

$$\lambda_1^p = \frac{\omega}{\omega_n\sqrt{\mu}}, \quad \lambda_1^a = \frac{\omega}{2\omega_n\sqrt{\mu}} \tag{6.9}$$

respectively, where ω_n is the greatest eigenvalue of the matrix C_1. For small μ we have

$$\lambda_1^p > 1, \quad \lambda_1^a > 1 \tag{6.10}$$

Hence, the number of eigenvalues in the range (0, 1) is $N(0) = N(\pi) = 0$. Consequently, in the case of condition (6.10) the stability condition (4.9) is satisfied, where the index of the stabilized system $q = N(0) = 0$. Taking into account the fact that $q(+0) = q + n = n$, $q(\pi) = 0$, we obtain that the index function $q(\varphi)$ decreases monotonically in the range $(0, \pi)$.

Hence, inequalities (6.10) serve as the conditions for parametric stabilization of Eq. (6.6). When μ and ω vary continuously, the strong stability breaks down, provided $\lambda_1^p(\mu, \omega) = 1$ and $\lambda_1^a(\mu, \omega) = 1$; the equalities determine the boundaries of the region of parametric stabilization.

Assuming $C_0 < 0$, we will consider this region in the μ, $\varepsilon = 1/\omega^2$ plane. As can be seen from Eq. (6.7), when ε increases the Hamiltonian decreases, and consequently the index function $q(\omega, \varepsilon)$ also decreases with ε. Since $q(\varphi, \varepsilon)$ decreases as φ in the range $(0, \pi)$, the stability breaks down when the interval $\Gamma_1 = [0, \varphi_1(\varepsilon))$ contracts to zero. Then the multiplier $\rho_1(\varepsilon) = \exp(i\varphi_i(\varepsilon)) = 1$, and hence on this boundary Eq. (6.7) has the periodic solution $\mathbf{x}(\tau) = \mathbf{x}(\tau + 2\pi)$ (i.e. $\lambda_1^p(\mu, \varepsilon) = 1$). Suppose $W(\tau, \mu, \varepsilon)$ is the matricant of the corresponding equation (1.1); the existence of a 2π-periodic solution implies that

$$\det\|W(2\pi, \mu, \varepsilon) - I_{2n}\| = 0 \tag{6.11}$$

Hence, for fixed μ the upper limit $\varepsilon_+(\mu)$ of the region of parametric stabilization is the least root of Eq. (6.11).

When ε decreases the index function $q(\varphi, \varepsilon)$ increases, and hence the stability of the stabilized system may break down, provided the interval $\Gamma_n = (\varphi_n(\varepsilon), \pi]$ contracts to a point π. Then the multiplier $\rho_n(\varepsilon) = -1$, and hence Eq. (6.6) has the antiperiodic solution $\mathbf{x}(\tau) = -\mathbf{x}(\tau + 2\pi)$ ($\lambda_1^a(\mu, \varepsilon) = 1$). Consequently, the lower boundary $\varepsilon_-(\mu)$ of the region of parametric stabilization is the least root of the equation.

$$\det\|W(4\pi, \mu, \varepsilon) - I_{2n}\| = 0 \tag{6.12}$$

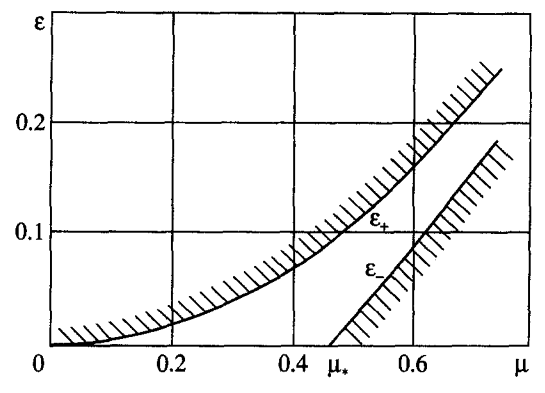

Fig. 7

For small μ Eq. (6.12) has no roots, and hence $\varepsilon_-(\mu) = 0$.

As an example we will consider the well-known problem of the parametric stabilization of the upper equilibrium position of a pendulum [14]. The corresponding equation reduces to the form

$$x'' + (-\varepsilon + \mu\cos\tau)x = 0; \quad \varepsilon = g/(l\omega^2), \quad \mu = a/l \tag{6.13}$$

where x, m and l are the angular coordinate, the mass and length of the pendulum, g is the acceleration due to gravity, and a and ω are the amplitude and frequency of the oscillations of the point of suspension.

In the case considered

$$W(\tau, \mu, \varepsilon) = \left\| \begin{matrix} x_1(\tau, \mu, \varepsilon) & x_2(\tau, \mu, \varepsilon) \\ x_1'(\tau, \mu, \varepsilon) & x_2'(\tau, \mu, \varepsilon) \end{matrix} \right\|$$

where $x_1(\tau, \mu, \varepsilon)$ and $x_2(\tau, \mu, \varepsilon)$ are the solutions of Eq. (6.13) with initial condition $x(0) = 1, x'(0) = 0$ and $x(0) = 0, x'(0) = 1$, respectively.

According to the results obtained, the boundaries of the region of parametric stabilization $\varepsilon_+(\mu)$ and $\varepsilon_-(\mu)$ are found from Eqs (6.11) and (6.12). These equations can be simplified by using the form of the periodic coefficient. Precisely because $\cos\tau$ is even, the periodic solutions are even or odd, i.e. they are identical with $x_1(\tau, \mu, \varepsilon)$ or $x_2(\tau, \mu, \varepsilon)$, respectively. Hence, when condition (6.11) is satisfied $x_1'(2\pi, \mu, \varepsilon) = 0$ or $x_2(2\pi, \mu, \varepsilon) = 0$. Taking into account the fact that $\cos\tau$ decreases in the range $(0, \pi)$ and increases in the range $(\pi, 2\pi)$, it can be shown that the first of these equations has the least root $\varepsilon(\mu)$. Finally, by virtue of the fact that $x_1(\tau, \mu, \varepsilon) = x_1(2\pi - \tau, \mu, \varepsilon)$ this is equivalent to the equation $x_1'(\pi, \mu, \varepsilon) = 0$, which also serves to define the upper boundary $\varepsilon_+(\mu)$ of the region of parametric stabilization. It can similarly be shown that the lower boundary $\varepsilon_-(\mu)$ is given by the equation $x_1'(2\pi, \mu, \varepsilon) = 0$.

In Fig. 7 we show the boundaries $\varepsilon_+(\mu)$ ad $\varepsilon_-(\mu)$ of the region of stability of Eq. (6.13), obtained using these equations. As can be seen, when $\mu < \mu_* \approx 0.46$ the upper position of the pendulum is stable for all $\omega > (\varepsilon_+(\mu))^{-1/2}$, whereas when $\mu > \mu_*$ an increase in ω initially stabilizes the system and then, when $(\omega > \varepsilon_-(\mu)^{-1/2})$, it is destabilizes the system.

REFERENCES

1. YAKUBOVICH, V. A. and STARZHINSKII, V. M., *Linear Differential Equations with Periodic Coefficients and their Applications*. Nauka, Moscow, 1972.
2. LYAPUNOV, A. M., The general problem of the stability of motion. In *LYAPUNOV, A.M., Selected Papers*, Vol. 2. Izd. Akad. Nauk SSSR, Moscow, 1956, 7–263.
3. POINCARÉ, H., *Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. 1. Gauthier-Villars, Paris, 1982.
4. KREIN, M. G., Fundamentals of the theory of λ-zones of stability of a canonical system of linear differential equations with periodic coefficients. In *Memory of A. A. Andronov*. Izd. Akad. Nauk SSSR, Moscow, 1955, 413–498.
5. GEL'FAND, I. M. and LIDSKII, V. B., The structure of the stability regions of linear canonical systems of differential equations with periodic coefficients. *Uspekhi Mat. Nauk*, 1995, **10**, 1, 3–40.
6. KREIN, M. G. and LYUBARSKII, G. Ya., The analytical properties of the multipliers of periodic canonical positive-type differential systems. *Izv. Akad. Nauk SSSR. Ser. Mat.*, 1962, **26**, 4, 549–572.
7. ZEVIN, A. A., The stability regions of linear canonical systems with periodic coefficients. *Prikl. Mat. Mekh.* 1998, **62**, 1, 41–48.

8. ZEVIN, A. A., A generalized index of a linear canonical system with periodic coefficients. *Dokl. Ross. Akad. Nauk*, 2002, 384, **1**, 7–10.
9. YAKUBOVICH, V. A., The convexity properties of the stability regions of linear Hamiltonian systems of differential equations with periodic coefficients. *Vestnik LGU. Ser. Mat. Mekh. Astronomii*, 1962, **13**, 3, 66–86.
10. BOLOTIN, V. V., *Dynamic Stability of Elastic Systems*. Gostekhizdat, Moscow, 1956.
11. YAKUBOVICH, V. A. and STARZHINSKII, V. M., *Parametric Resonance in Linear Systems*. Nauka, Moscow, 1987.
12. ZEVIN, A. A., Symmetrization of functionals and its applications in mechanics problems. *Izv. Akad. Nauk SSSR. MTT*, 1991, 5, 109–118.
13. ZEVIN, A. A., Analysis of stability and instability regions in parametrically excited Hamiltonian systems. *Nonlinear Dynamics*, 1997, **12**, 4, 327–341.
14. BOGOLYUBOV, N. N. and MITROPOL'SKII, Yu. A., *Asymptotic Methods in the Theory of Non-linear Oscillations*. Nauka, Moscow, 1974.
15. CHELOMEI, V. N., The possibility of increasing the stability of elastic systems using vibrations. *Dokl. Akad. Nauk SSSR*, 1956, **110**, 3, 345–347.

Translated by R.C.G.